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A Comparative Study of Multidimensional Sampling Plans

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Summary

It has been pointed out by Bhat and Kulkarni [1] that the Multinomial sampling plan (MSP) and Inverse multinomial sampling plan (IMSP) are multidimentsional efficient sampling plans in DeGroot's [2] sense. A comparative study of these two multidimensional sampling plans has been made for estimating $g(p) = \sum_{j=1}^{r} \lambda_j p_j$ which is a linear function of multinomial proportions, $p = (p_1, p_2, \ldots, p_r)$, where $p_j > 0$, $j = 1, 2, \ldots r+1$ and $\sum_{j=1}^{r+1} p_j = 1$ and it has been assertained that IMSP is also an useful sampling plan for estimating multinomial proportions or a linear function of them.

Key words : Bernoulli population, Multinomial sampling plan, Inverse multinomial sampling plan, Information inequality, Maximum likelihood estimator, Efficient estimator, Relative efficiency.

Introduction

Consider a multivariate random variable $y = (e_1, e_2, \ldots, e_{r+1})$, where $e_j, j = 1, 2, \ldots, (r+1)$ are either zero or one such that $\sum_{j=1}^{r+1} e_j = 1$.

$$P(\text{y is a vector with j-th place unity}) = p_j > 0, (j = 1, 2, \ldots, (r+1)).$$

i.e.
$$P(y) = \prod_{t=1}^{r+1} p_t^{e_t} \tag{1.1}$$

Let $y_1, y_2, \ldots$ be a sequence of observations on random variable y and $x = \sum_{i=1}^{N} y_i = (x_1, x_2, \ldots, x_{r+1})$ be the sum of N independent observations. These observations can be represented by a lattice path in (r + 1)- dimensions starting from the origin and moving at the j-th step, one unit along or parallel to Z_k-axis,

($k = 1, 2, \ldots, (r + 1)$) if the k-th place of y_j is unity. The $(r+1)$-dimensional nomenclature can be viewed as a multidimensional extension of the two-dimensional one due to Girshick, et al [3]. The sample size N, when the process terminates at the boundary point $x(\alpha) = [x_1(\alpha), x_2(\alpha), \ldots, x_{r+1}(\alpha)]$, is the sum $x_1(\alpha) + \ldots + x_{r+1}(\alpha)$.

The probability of reaching the boundary point $x(\alpha)$ is

$$P(\theta, \alpha, p) = \Pi_\theta(\alpha) \prod_{t=1}^{r+1} p_t^{x_t}, \tag{1.2}$$

where $\Pi_\theta(\alpha)$ is the number of paths leading to the boundary point $B = \{x(\alpha) : x_i(\alpha) \geq 0, i = 1, 2, \ldots, (r+1)\}$ and θ is the parameter of the boundary points.

A sampling plan with boundary B is said to be closed if

$$\sum_{\alpha \in B} \Pi(\alpha) \prod_{t=1}^{r+1} p_t^{x_t} = 1 \tag{1.3}$$

for all $p_t \in (0, 1), t = 1, 2, \ldots, (r+1)$ and $\sum_{t=1}^{r+1} p_t = 1$. An estimator $T(\alpha)$ is a real-valued function defined on B such that

$$E[T(\alpha)] = \sum_{\alpha \in B} T(\alpha) \Pi(\alpha) \prod_{t=1}^{r+1} p_t^{x_t} \tag{1.4}$$

is absolutely convergent. It can easily be seen that $x(\alpha)$, the boundary points of multidimensional sampling plans, are complete sufficient statistics for the family of distributions generated over the boundary points due to a given stopping rule.

The parameters p_j ($j = 1, 2, \ldots, (r+1)$) can be estimated using the method of maximum likelihood estimation. It is seen in section (3.1) that IMSP is better than MSP, (or the converse) for estimating a linear function of $p = (p_i, \ldots, p_r)$ if the average sample size of the corresponding sampling plan is larger (or smaller).

Secondly, the unique minimum variance unbiased estimators of $p = (p_1, p_2, \ldots, p_{r+1})$ have been calculated in respective sampling plans and the criterion of comparison of these two sampling plans has been developed as $\| (c/p_{r+1}) V_I - n V_M \|$, where (c/p_{r+1}) is the ASN of the IMSP, V_I is the variance- covariance matrix of the estimators of p in IMSP and n is the ASN of MSP, V_M is the variance-covariance matrix of the estimators of p in MSP. It is observed in section (3.1) that IMSP is a competitor to MSP for estimating a linear function of $p = (p_1, p_2, \ldots, p_r)$.

Thirdly, to compare the sampling plans for estimating the proportions $p_1, p_2, \ldots, p_{r+1}$, let

$$e_j(p) = \frac{E(N_I)\, V_I(f_j)}{E_M(N)\, V_M(f_j)} \tag{1.5}$$

be the measure of relative efficiency of an estimator of $g(P) = p_j\ (j = 1, 2, \ldots, r+1)$ in MSP to IMSP where f_j stands for the estimator of $g(p) = p_j$, 'I' stands for IMSP and 'M' stands for MSP. Let $R_j(p) = [e_j(p) - 1]$. We have derived the bounds for $R_j(p)$ for comparing IMSP with MSP in estimating $p_j\ (j = 1, 2, \ldots, r+1)$ in section 3.2.

2. *Multinomial and inverse Multinomial Sampling Plans and Unique Unbiased Estimation of* $p = (p_1, p_2, \ldots, p_r)$.

2.1 *Multinomial Sampling Plan (MSP)*

Consider a population which is described by the law (1.1). Let $y_1, y_2, \ldots$ be a sequence of observations on the random variable y and we stop taking observations with the help of a stopping rule. According to the MSP, the stopping rule is to stop after taking n (a predetermined number of) observations. Let $x = \sum_{i=1}^{n} y_i = (x_1, \ldots, x_{r+1})$, hence, $\sum_{i=1}^{r+1} x_i = n$. The boundary points of the plan are given by

$$B_M = [(x_1, x_2, \ldots, x_{(r+1)}) : x_i \geq 0 \text{ for } i = 1, 2, \ldots, (r+1) \text{ and } \sum_{i=1}^{r+1} x_i = n].$$

2.2 *The Inverse Multinomial Sampling Plan (IMSP)*

According to this plan, observations are taken in a sequence, until a predetermined number c of observations fall into a given class. Let us, for example, assume that we take observations just until $x_{r+1} = c$ have been observed. The probability that the process terminates after reaching a boundary point of

$$B_I = [(x_1, x_2, \ldots, x_r, c) : x_i \geq 0 \text{ for } i = 1, 2, \ldots, r \text{ and } c > 0]$$

is given as

$$P_I(\alpha) = \Pi_c(\alpha)\, p_{r+1}^{c} \prod_{t=1}^{r} p_t^{x_t} \tag{2.2.1}$$

where $\quad 0 < p_t < 1 \text{ for } t = 1, 2, \ldots, r+1$

and $\quad \sum_{t=1}^{r+1} p_t = 1$ and $\Pi_c(\alpha) = \dfrac{\left(c - 1 + \sum_{i=1}^{r} x_i\right)!}{(c-1)!\, \prod_{i=1}^{r} x_i!}.$

2.3 *Minimum Variance Unbiased Estimators of Parameters and Their Variance and Covariance Matrices*

It is well known that the unique minimum variance unbiased estimators of $p_i(i = 1, 2, \ldots, r+1)$ in MSP are

$$\hat{p}_{i(1)} = (x_i/n), \text{ for } i = 1, 2, \ldots, r+1 \tag{2.3.1}$$

and the unique minimum variance unbiased estimators of p_i for $i = 1, 2, \ldots, r+1$ in IMSP are

$$\hat{p}_{i(2)} = \frac{x_i}{c - 1 + \sum_{j=1}^{r} x_j}, \text{ for } i = 1, 2, \ldots, r \tag{2.3.2}$$

and

$$\hat{p}_{r+1(2)} = \frac{c-1}{c - 1 + \sum_{i=1}^{r} x_i} \tag{2.3.3}$$

as they are functions of complete sufficient statistics.

Assume that $\hat{p}_{(j)} = (\hat{p}_{1(j)}, \ldots, \hat{p}_{r(j)})$ be the vector of independent estimators of $p = (p_1, \ldots, p_r)$ for j-th plan $(j = 1, 2)$ as $\hat{p}_{r+1(j)} = 1 - \sum_{i=1}^{r} \hat{p}_{i(j)}$ $(j = 1, 2)$. Here 1 stands for MSP and 2 for IMSP. The exact variance and covariance matrices of the estimators $\hat{p}_{(j)}$ in respective plans are given by

$$V_j = (v_{il}^{(j)}, i, l = 1, 2, \ldots, r) \text{ and } j = 1, 2 \tag{2.3.4}$$

where $V_{il}^{(j)}$ are given by (2.3.5) and (2.3.10).

The variance and covariance matrices for the estimators of p in MSP can be seen from Khirsagar [4] and we have derived it for IMSP in this section.

(a) The variance-covariance matrix of the estimators $\hat{p}_{(1)} = (\hat{p}_{i(1)}, \ldots, \hat{p}_{r(1)})$ in MSP.

It is known that,

$$V_{ii}^{(1)} = V(\hat{p}_{i(1)}) = \frac{p_i(1-p_i)}{n}, \text{ for } i = 1, 2, \ldots, (r+1)$$

$$V_{ik}^{(1)} = \text{Cov}(\hat{p}_{i(1)}, \hat{p}_{k(1)}) = -\frac{p_i p_k}{n}, i \neq k = 1, \ldots, (r+1) \tag{2.3.5}$$

and $$V_1 = (V_{ik}^{(1)}) = \sum\nolimits_1^{-1} = (1/n)\,[\text{diag}(p_1, p_2, \ldots, p_r) - p'p] \tag{2.3.6}$$

with $p = (p_1, p_2, \ldots, p_r)$.

(b) The variance-covariance matrix of the estimators $\hat{p}_{(2)} = (\hat{p}_{1(2)}, \hat{p}_{2(2)}, \ldots, \hat{p}_{r(2)})$.

The Variance-covariance matrix

$$V_2 = (V_{il}^{(2)}; i, l = 1, 2, \ldots, r) \tag{2.3.7}$$

is computed as follows.

It can be easily seen in (2.2.1) that for given $y = \sum_{i=1}^{r} x_i$

(i) $$E(x_i \mid y) = \frac{y p_i}{1 - p_{r+1}} \text{ for } i = 1, 2, \ldots, r$$

(ii) $$E(x_i x_k \mid y) = \frac{y(y-1)\, p_i\, p_k}{(1-p_{r+1})^2} \text{ for } i \neq k \; (i, k = 1, 2, 3, \ldots, r) \text{ and}$$

(iii) $$E(x_i^2 \mid y) = \frac{y(y-1)p_i^2}{(1-p_{r+1})^2} + \frac{y p_i}{1-p_{r+1}} \qquad i = 1, 2, \ldots, r \tag{2.3.8}$$

The marginal distribution of y is negative binomial, given by

$$\frac{(c-1+y)!}{(c-1)!\,y!}\,p_{r+1}^{c}\,(1-p_{r+1})^{y},\ \text{for } y = 0, 1, 2, \ldots \tag{2.3.9}$$

Using (2.3.9) and the conditional arguments (2.3.8) the variance and covariance matrix of $\hat{p}_{(2)} = (\hat{p}_{1(2)}, \ldots, \hat{p}_{r(2)})$ can be computed as

$$V_2 = (v_{ik}^{(2)}), \tag{2.3.10}$$

where $$V_{ii}^{(2)} = cp_i^2\, f(c+2) + p_i\, p_{r+1}\, f(c+1) - p_i^2 \quad \text{for } i = 1, 2, \ldots, r. \tag{2.3.11}$$

$$V_{ik}^{(2)} = cp_i p_k\, f(c+2) - p_i\, p_k,\ \text{for } i \neq k = 1, \ldots, r \tag{2.3.12}$$

and $$f(c) = (c-2)!\sum_{j=0}^{\infty}\frac{j!(1-p_{r+1})^{j}}{(c-1+j)!} \tag{2.3.13}$$

Further, we may derive, along the same line,

$$\operatorname{Var}(\hat{p}_{r+1(2)}) = (c-1)\,p_{r+1}^2\, f(c) - p_{r+1}^2 \tag{2.3.14}$$

$$\operatorname{Cov}(\hat{p}_{i(2)}, \hat{p}_{r+1(2)}) = (c-1)\,p_{r+1}p_i\, f(c+1) - p_{r+1}p_i \tag{2.3.15}$$

The values of f(c) for different values of p_{r+1} and c have been computed (see Table I of the appendix) to calculate the variance and covariance matrix V_2.

3. *Comparative Study of Multidimensional Sampling Plans for Estimating a Linear Function of Proportions* $p = (p_1, p_2, \ldots, p_r)$ *or Multinomial Proportions*

It is known that the variance of the estimator decreases with the ASN when one sampling plan is under consideration. The unbiasedness and minimization of the variance provide satisfactory consideration for obtaining a point estimator for a sampling plan. But when two sampling plans are to be compared, the role of ASN creeps in from the background as it is usually expected that the variance will decrease with the increase of sample size. It is therefore proposed to measure the efficiency of a plan in estimating a function of a parameter by the precision of the estimator per unit ASN. In practice, it will be more convenient to compute and compare the reciprocal of this quantity which is the product of the variance of the estimator and the ASN of the plan. A plan is best if this product is least. Thus E(N)V(f) is an effective optimal criteria of an estimator for simultaneous minimization of ASN and the variance of an estimator.

Let $\hat{g}(p) = \sum_{i=1}^{r} \lambda_i \hat{p}_{i(j)}$ be the estimator of $g(p) = \sum_{i=1}^{r} \lambda_i p_i$, for the j-th plan (j = 1, 2). Then $V_j(\hat{g}(p)) = \lambda V_j \lambda'$, for j = 1, 2, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Hence , $n_j V_j(\hat{g}(p)) = \lambda[n_j V_j] \lambda'$, for j = 1, 2. Therefore , $n_2 V_2(\hat{g}(p))(-n_1 V_1(\hat{g}(p)) = \lambda[n_2 V_2 - n_1 V_1]\lambda'$. Now, the IMSP is preferred to MSP if $\| (c/p_{r+1})V_2 - nV_1 \|$ is negative. We have derived here the expressions for each element b_{ij} $(i, j = 1, 2, \ldots, r)$ of the determinant which are easily computable, where $|(b_{ij})| = \| c/p_{r+1})V_2 - nV_1 \|$. It is noted that all the diagonal elements of the determinant may be positive where as all off diagonal elements may be negative for some values of c and p. however, it is not practicable to express the elements of $|(b_{ij})|$ in a simpler form so as to know the exact values of c and p for which IMSP is better than MSP. But the value of $|(b_{ij})|$ (which can be computed for a given value of c and p using the derived expressions and with the help of given Table) helps to know the behavior of the plans in a particular case. Here, the value of the determinant is computed for c = 2 and 5 and for different values of p_i, $i = 1, 2$ when $r = 2$. It has been seen that in all these cases, the diagonal elements are positive and off diagonal elements are negative. It is also seen that the values of the determinants for c = 2 and 5 are close to zero. From this it is quite clear that IMSP is a competitor to MSP for estimating a linear function of $(p_1, p_2, \ldots, p_r)$.

3.1 *Computation of the criteria of comparison*

The elements of the determinant $|(b_{ij})|$ are derived in a computable form, as follows.

(1) Derivation of b_{ii}, for i = 1, 2, 3, . . ., r.

$$(c/p_{r+1})V_{ii}^{(2)} = (c^2/p_{r+1})\, p_i^2\, f(c+2) + cp_i\, f(c+1) - (cp_i^2/p_{r+i})$$

$$= \frac{c^2 p_i^2}{p_{r+1}} c! \sum_{j=0}^{\infty} \frac{j!(1-p_{r+1})^j}{(c+1+j)!} + c\, p_i\, (c-1)! \sum_{j=0}^{\infty} \frac{j!(1-p_{r+1})^j}{(c+j)!}$$

$$-(c\, p_i^2/p_{r+1}).$$

Substituting for f(c), given by (2.3.13).

Similarly, $n V_{ii}^{(1)} = p_i (1 - p_i)$, or i = 1, 2, . . . , r. [using (2.3.5)]

Therefore, $b_{ii} = (c/p_{(r)} + 1)\, V_{ii}^{(2)} - n\, V_{ii}^{(1)}$, for i = 1, 2, . . . , r

$$= \frac{c^2 p_i^2}{p_{r+1}} c\,! \sum_{j=0}^{\infty} \frac{j\,!(1-p_{r+1})^j}{(c+1+j)\,!} + p_i\, c\,! \sum_{j=0}^{\infty} \frac{j\,!(1-p_{r+1})^j}{(c+j)\,!}$$

$$- \frac{c\, p_i^2}{p_{r+1}} - p_i\,(1-p_i)$$

$$= \left(\frac{p_i^2}{p_{r+1}}\right)\left[p_{r+1} - \frac{c}{c+1}\right]$$

$$+ \left(\frac{p_i}{p_{r+1}}\right) c\,! \sum_{j=1}^{\infty} [c^2 p_i + p_{r+1}\,(c+1+j)] \frac{j\,!\,(1-p_{r+1})^j}{(c+1+j)\,!} \tag{3.1.1}$$

On simplification,

$$b_{ii} = \frac{p_i^2}{p_i\,(c+1)(c+2)}\,[(3c+2)p_{r+1} - 2c] + \frac{p_i\,(1-p_{r+1})}{(c+1)} +$$

$$\left(\frac{p_i}{p_{r+1}}\right) c\,! \sum_{j=2}^{\infty} [c^2 p_i + p_{r+1}\,(c+1+j)] \frac{j\,!\,(1-p_{r+1})^j}{(c+1+j)\,!} \tag{3.1.2}$$

for $i = 1, 2, \ldots, r$.

(2) Derivation of b_{ik} for $i \neq k = 1, 2, \ldots, r$.

$$\frac{c}{p_{r+1}} V_{ik}^{(2)} = \frac{c\, p_i^2\, p_k}{p_{r+1}} c\,! \sum_{j=0}^{\infty} \frac{j\,!(1-p_{r+1})^j}{(c+1+j)\,!} - \frac{c p_i\, p_k}{p_{r+1}}$$

substituting the expression (2.3.13) of f(c).

Again, $n\,V_{ik}^{(1)} = -p_i\, p_k$ for $i \neq k = 1, 2, \ldots, r$.

Hence, $b_{ik} = \dfrac{c}{p_{r+1}} V_{ik}^{(2)} - n\, V_{ik}^{(1)}$, for $i \neq k = 1, 2, \ldots, r$.

$$= \frac{c^2 p_i p_k}{p_{r+1}} c\,! \sum_{j=0}^{\infty} \frac{j\,!(1-p_{r+1})^j}{(c+1+j)\,!} - \frac{c p_i p_k}{p_{r+1}} + p_i\, p_k$$

$$= -[c - p_{r+1}] \frac{p_i\, p_k}{p_{r+1}} + \frac{c_2\, p_i\, p_k}{p_{r+1}} c\,! \sum_{j=1}^{\infty} \frac{j!\,(1-p_{r+1})^j}{(c+1+j)\,!} \tag{3.1.3}$$

which may be negative for large values of c.

We have calculated below the values of the determinant for different values of c and p and observed that IMSP is a competitor to MSP for estimating a linear function of $(p_1, p_2, \ldots, p_r)$. It can also be seen that all the elements of (b_{ij}) $i, j = 1, 2, \ldots, r$ converges rapidly and hence it is easy to compute with the help of a computer.

CASE-I : $c = 2$

i (a) $p_1 = .7, p_2 = .2$ and $p_3 = .1$

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0223 & -0{\cdot}0102 \\ -0{\cdot}0007 & 0{\cdot}0136 \end{pmatrix}$$

The value of $|(b_{ij})| = 2(10)^{-4}$.

i (b) $p_1 = .3, p_2 = .2$ and $p_3 = .5$

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0716 & -0{\cdot}0136 \\ -0{\cdot}0136 & 0{\cdot}0523 \end{pmatrix}$$

The value of $|(b_{ij})| = 4(10)^{-3}$.

i (c) $p_1 = .07, p_2 = .03$ and $p_3 = .9$

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0311 & -0{\cdot}0007 \\ -0{\cdot}0007 & 0{\cdot}0137 \end{pmatrix}$$

The value of $|(b_{ij})| = 4(10)^{-4}$.

CASE-II : $c = 5$

ii (a) $p_1 = .7, p_2 = .2$ and $p_3 = .1$.

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0050 & -0{\cdot}0033 \\ -0{\cdot}0033 & 0{\cdot}0039 \end{pmatrix}$$

The value of $|(b_{ij})| = 9(10)^{-6}$.

ii (b) $p_1 = .3, p_2 = .2$ and $p_3 = .5$.

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0241 & -0{\cdot}0059 \\ -0{\cdot}0059 & 0{\cdot}0180 \end{pmatrix}$$

The value of $|(b_{ij})| = 4(10)^{-4}$.

ii(c) $p_1 = .07, p_2 = .03$ and $p_3 = .9$.

$$(b_{ij}) = \begin{pmatrix} 0{\cdot}0121 & -0{\cdot}0003 \\ -0{\cdot}0003 & 0{\cdot}0054 \end{pmatrix}$$

The value of $|(b_{ij})| = 7(10)^{-5}$.

3.2 *Comparison of the Plans for Estimating p (i = 1, . . ., r + 1)*

The bounds for $R_i(c, p) = \dfrac{n_2 V(\hat{p}_{i(2)})}{n_1 V(\hat{p}_{i(1)})} - 1 = \dfrac{b_{ii}}{n_1 V(\hat{p}_{i(1)})}$,

$i = 1, 2, \ldots, (r + 1)$ have been derived in this section. It is seen that the bounds for $R_{r+1}(c, p)$ is positive but the bounds of $R_i(c,p)$ $(i = 1, 2, \ldots, r)$ may be negative in certain range of p and large values of c.

Theorem 3.2.1 : (A) For estimating p $(i = 1, 2, \ldots, r)$

$$\frac{1}{p_{r+1}}\left(\frac{p_i}{1-p_i}\right)\frac{\{(3c+2)p_{r+1}-2c\}}{(c+1)(c+2)} < R_i(c, p) <$$

$$\frac{1}{p_{r+1}}\left(\frac{p_i}{1-p_i}\right)\left[p_{r+1} - \frac{2c}{(c+1(c+2)}\right] + \frac{1}{(c+1)(1-p_i)}.$$

(B) For estimating p_{r+1}

$$\frac{2(1-p_{r+1})}{(c+1)} \leq R_{r+1}(c, p) \leq \frac{2(1-p_{r+1})}{(c-2)} \text{ for } c \geq 3.$$

Proof : (A) From the relations (3.1.2) and (2.3.5), we can have,

$$R_i(c, p) = \frac{b_{ii}}{n\, V(\hat{p}_{i(1)})}$$

$$= \frac{p_i}{p_{r+1}(1-p_i)}\frac{\{(3c+2)p_{r+1}-2c\}}{(c+1)(c+2)} + \frac{(1-p_{r+1})}{(c+1)(1-p_i)}$$

$$+ c!\sum_{j=2}^{\infty}\frac{[c\,p_i + p_{r+1}(c+1+j)]}{p_{r+1}(1-p_i)}\frac{j!(1-p_{r+1})^j}{(c+1+j)!} \quad \text{for } i = 1, 2, \ldots, r.$$

or, $$R_i(c, p) > \frac{p_i}{p_{r+1}(1-p_i)}\frac{\{(3c+2)p_{r+1}-2c\}}{(c+1)(c+2)} \tag{3.2.1}$$

Again, using (3.1.1) and (2.3.5) we can deduce as

$$R_i(c, p) = \frac{p_i}{p_{r+1}(1-p_i)}\left[p_{r+1} - \frac{c}{c+1}\right]$$

$$+ \frac{c\,!}{p_{r+1}(1-p_i)} \sum_{j=1}^{\infty} [c^2 p_i + p_{r+1}(c+1+j)] \frac{j\,!(1-p_{r+1})^j}{(c+1+j)\,!}$$

$$+ \frac{c\,!}{(1-p_i)} \sum_{j=1}^{\infty} \frac{j\,!(1\ p_{r+1})^j}{(c+j)\,!}$$

or, $$R_i(c, p) \leq \frac{p_i}{p_{r+1}(1-p_i)}\left[p_{r+1} - \frac{c}{c+1}\right] + \frac{c^2\, c\,! p_i}{p_{r+1}(1-p)} \sum_{y=0}^{\infty} \frac{(y+1)\,!}{(y+c+2)\,!}$$

$$+ \frac{c\,!}{(1-p_i)} \sum_{y=0}^{\infty} \frac{(y+1)\,!}{(y+c+1)\,!}$$

Now, using the identity,

$$\sum_{y=0}^{\infty} \frac{(y+j)\,!}{(y+k)\,!} = \frac{j\,!}{(k-j-1)(k-1)\,!} \text{ for } k > (j+1) \tag{3.2.2}$$

We will have,

$$R_i(c, p) \leq \frac{p_i}{p_{r+1}(1-p_i)}\left[p_{r+1} - \frac{2c}{(c+1)(c+2)}\right] + \frac{1}{(c+1)(1-p_i)} \tag{3.2.3}$$

The part (A) of the theorem is proved combining the results (3.2.1) and (3.2.3).

(B) It can be simplified as

$$R_{r+1}(c, p) = \frac{b_{(r+1)(r+1)}}{nV(\hat{p}_{r+1(1)})} = c\,! \sum_{j=2}^{\infty} \frac{j\,!(1-p_{r+1})^{j-1}}{(c+1+j)}$$

Using the relations (2.3.5) and (2.3.14).

$$\frac{2(1-p_{r+1})}{c+1} + c! \sum_{y=2}^{\infty} \frac{(y+1)\,!\,(1-p_{r+1})^y}{(c+y)\,!}$$

Hence, $$R_{r+1}(c, p) > \frac{2(1-p_{r+1})}{(c+1)} \tag{3.2.4}$$

Again, using the identity (3.2.2) it can be verified that

$$R_{r+1}(c, p) < \frac{2(1 - p_{r+1})}{(c - 2)} \tag{3.2.5}$$

The part (B) of the theorem is proved combining the results (3.2.4) and (3.2.5).

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APPENDIX

The Table-I has been prepared for computing f(c) for different values of c and p_{r+1} to compute the variance and covariance matrix (2.3.10).

$$\textbf{Table I.}\ \ \text{SUM} = f(c) = (c-2)!\sum_{j=0}^{\infty}\frac{j!\,(1-p_{r+1})^j}{(c-1+j)!}\,.$$

c	p	SUM	c	p	SUM	c	p	SUM
2	.01	4.553994	8		0.160566	5		0.273242
3		0.962644	9		0.138430	6		0.215137
4		0.495317	10		0.121643	7		0.177295
5		0.331697	2	.3	1.719961	8		0.150725
6		0.249175	3		0.691445	9		0.131056
7		0.199503	4		0.417952	10		0.115916
8		0.166335	5		0.297068	2	.7	0.104976
9		0.142620	6		0.229828	3		0.095025
10		0.124822	7		0.187216	4		0.086794
2	.05	3.153350	8		0.157859	5		0.079873
3		0.886663	9		0.136428	6		0.210969
4		0.479649	10		0.120103	7		0.174406
5		0.325634	2	.4	1.527151	8		0.148608
6		0.246019	3		0.648566	9		0.12943
7		0.197578	4		0.400956	10		0.11464
8		0.165040	5		0.288252	2	.8	1.115717
9		0.141690	6		0.224499	3		0.537129
10		0.124122	7		0.183667	4		0.351484
2	.1	2.558427	8		0.155333	5		0.260730
3		0.826841	9		0.134540	6		0.207079
4		0.463684	10		0.118640	7		0.171684
5		0.318850	2	.5	1.386295	8		0.146598
6		0.242350	3		0.613706	9		0.131056
7		0.195294	4		0.386294	10		0.115916
8		0.163486	5		0.280372	2	.9	1.053605
9		0.140565	6		0.219628	3		0.517554
10		0.123272	7		0.180372	4		0.342018
2	.2	2.011797	8		0.152961	5		0.255174
3		0.747051	9		0.132753	6		0.203432
4		0.438237	10		0.117247	7		0.169109
5		0.307107	2	.6	1.277064	8		0.144684
6		0.235723	3		0.584404	9		0.129439
7		0.191069	4		0.373394	10		0.114641